

On the regular-geometric-figure solution to the N-body problem

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Abstract. The regular-geometric-figure solution to the N -body problem is presented in a very simple way. The Newtonian formalism is used without resorting to a more involved rotating coordinate system. Those configurations occur for other kinds of interactions beyond the gravitacional ones for some special values of the parameters of the forces. For the harmonic oscillator, in particular, it is shown that the N -body problem is reduced to N one-body problems.

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Despite the efforts of mathematicians and physicists over more than two centuries of research the general problem of N mutually interacting bodies moving according to Newton's laws, for $N > 2$, has never been solved exactly. The two-body problem subject to forces which depend on the relative vector positions can be reduced to two one-body problems, one of which describes the motion of the center of mass and the other one the relative motion. $N = 3$ is the smallest value of N that turns the N -body problem unsolvable in the general case. However, under special assumptions about the type of motion and interaction, analytical solutions of the N -body problem can be found.

In the case of the three-body problem with gravitational interactions some special solutions are usually presented in the textbooks on classical mechanics. In the so-called restricted three-body problem two heavy bodies move about the common center of mass whereas a third light body moves in the same plane [1]-[3]. In the so-called Lagrange's case the three bodies are at the vertices of an equilateral triangle in every part of the motion, which rotates about an axis perpendicular to the plane of the bodies while it changes in size [4]-[7]. There is still another special solution for three bodies interacting by gravitational forces known as Euler's case. In that last case the bodies move along the same straight line during all the motion [8]-[10]. Another special solution is that of N bodies of similar masses subject to similar forces moving on the vertices of a regular N -sided polygon [11]. All of these special solutions are of great pedagogical importance since they give solutions of a problem unsolvable in the general case. Nevertheless the resolution of Lagrange's case, as presented by textbook authors, resorts to a rotating coordinate system requiring an extensive calculation, and consequently to a weakening of the pedagogical attractiveness.

In a previous work [12] Lagrange's case has been presented in an alternative and more general way, permitting that it can be easily approached immediately after the presentation of the two-body problem. In that paper the equilateral-triangle solutions were obtained for interactions which go beyond the gravitational ones. Essén, in a recent paper to this journal [13] (homonym to that one in Ref. [12]), approached this very same problem using the Lagrangian formalism restricting himself to gravitational interactions and presenting an extension to the N -body problem. Encouraged by the results obtained in [13] and following the same steps of Ref. [12], in this paper we reduce the N -body problem to N one-body problems in the case of more general central two-body interactions, such that the total force on each body is directed towards, or from the center of mass of the N -body system. As a by-product, for gravitational interactions we obtain an extension of the Lagrange case. We obtain that the N bodies are on the vertices of a non-rigid (in general) regular geometric figure throughout the motion. For other kind of interactions, this is also possible if certain conditions are satisfied by the parameters characterizing the intensities of the forces.

Let us consider N bodies (treated as particles) with masses m_i ($i = 1, \dots, N$) located by the vectors \mathbf{r}_i . It is supposed that the forces are pairwise, directly proportional to an arbitrary exponent of the distance between the bodies and directed along the line connecting them. Thus the resulting force acting on the i -th body due to the other

$N - 1$ bodies can be written as

$$\mathbf{F}_i = - \sum_{\substack{j=1 \\ (j \neq i)}}^N K_{ij} \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^n} \quad (1)$$

where K_{ij} ($K_{ii} = 0$ and $K_{ij} = K_{ji}$, according to Newton's third law) is the proportionality constant, r_{ij} is the distance between the bodies i and j , and the exponent n is a real number. Included within the possibilities allowed by (1) are some familiar forms of interactions such as the gravitational and the harmonic oscillation ones. For the gravitational case we have $n = 3$ and $K_{ij} = Gm_i m_j$, where G is the universal gravitation constant, whereas for a harmonic-oscillator-type interaction $n = 0$.

The N -body problem is not subject to external forces and consequently the center of mass of the system has no acceleration and, for the sake of simplicity, it is considered as being in rest. The position of the i -th body from the center of mass of the system \mathbf{r}'_i is related to \mathbf{r}_i by

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R} \quad (2)$$

where \mathbf{R} is the vector position of the center of mass. Therefore in the center-of-mass system frame the force acting on the i -th body takes the form

$$\mathbf{F}_i = - \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{K_{ij}}{r_{ij}^n} \mathbf{r}'_i + \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{K_{ij}}{r_{ij}^n} \mathbf{r}'_j \quad (3)$$

This force can also be written in the form

$$\mathbf{F}_i = - \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{K_{ij}}{r_{ij}^n} \mathbf{r}'_i + \sum_{\substack{j=1 \\ (j \neq i) \\ (j \neq k)}}^N \frac{K_{ij}}{r_{ij}^n} \mathbf{r}'_j + \frac{K_{ik}}{r_{ik}^n} \mathbf{r}'_k \quad (k \neq i) \quad (4)$$

Using the fact that

$$\sum_{i=1}^N m_i \mathbf{r}'_i = \mathbf{0} \quad (5)$$

by the definition of the center of mass frame, we can write

$$\mathbf{r}'_k = - \frac{1}{m_k} \sum_{\substack{s=1 \\ (s \neq k)}}^N m_s \mathbf{r}'_s \quad (6)$$

After substituting (6) in the last term of (4) and rearranging all the terms we get

$$\mathbf{F}_i = - \left(\sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{K_{ij}}{r_{ij}^n} + \frac{K_{ik}}{r_{ik}^n} \frac{m_i}{m_k} \right) \mathbf{r}'_i + \sum_{\substack{j=1 \\ (j \neq i) \\ (j \neq k)}}^N \left(\frac{K_{ij}}{r_{ij}^n} - \frac{K_{ik}}{r_{ik}^n} \frac{m_j}{m_k} \right) \mathbf{r}'_j \quad (k \neq i) \quad (7)$$

If the following condition is satisfied

$$\frac{\frac{K_{ij}}{r_{ij}^n}}{m_i m_j} = \frac{\frac{K_{ik}}{r_{ik}^n}}{m_i m_k} \quad (j \neq i, k \neq i) \quad (8)$$

i. e.,

$$\frac{K_{ij}}{r_{ij}^n} = \lambda m_i m_j \quad (9)$$

for some proportionality factor λ characteristic of the interactions as well as the geometrical configuration of the system, the resulting force acting on each particle will be along the line connecting it to the common center of mass of the N -body system. When the parameters K_{ij} and masses meet the requirement of (8) we have

$$\mathbf{F}_i = -\frac{K_{ij}^{eff}}{r_{ij}^n} \mathbf{r}'_i \quad (j \neq i) \quad (10)$$

where

$$K_{ij}^{eff} = \lambda m m_i \quad (11)$$

is the effective proportionality constant and $m = \sum_{i=1}^N m_i$ is the total mass of the system. Therefore the N -body problem reduces to N coupled problems of bodies exposed to central forces, with the common center of force located in the center of mass of the system. The coupling occurs by the presence of r_{ij} in (10).

As a particular case one has the gravitational one, for the three-body problem with the three bodies moving on the vertices of an equilateral triangle and in addition for four bodies on the vertices of a regular tetrahedron ($r_{ij} = r_{ik}$, for all i, j and k with $i \neq j$ and $i \neq k$). One should also note that there is no reason for maintaining $r_{ij} = \text{const}$, *i. e.*, a rigid regular geometric figure, it is just enough that any side or edge be congruent to any other during all the time in order to maintain the regular character of the geometric figure. Thus, it means that λ may vary with time. In other words, the regular geometric figures may change in size with their sides or edges expanding or contracting at the same rhythm, as in the original Lagrange's case (three gravitating bodies, $r_{ij}^3 = G/\lambda$), and in the case of the tetrahedron as well. As a matter of fact the equilateral-triangle figure may also remain rigid while the bodies rotate about an axis perpendicular to the plane with a common angular velocity, and in this situation the bodies move on circles centered at the center of mass of the system, which coincides with the common center of force. For a regular-tetrahedral figure things are indeed different because one should have four bodies on the vertices of a rigid tetrahedron moving on the great circles of concentric spheres, and this is impossible even in the case of equal masses when one should have just one sphere.

The equilateral-triangle and the regular-tetrahedral solutions also occur in a more general situation where

$$\frac{K_{ij}}{K_{ik}} = \frac{m_j}{m_k} \quad (12)$$

an n -independent condition, equivalent to $K_{ij} = \alpha_i m_j$, where α_i is a constant.

It is worthwhile to note that the constraint (12) implies that the N -body problem reduces to N one-body problems only in the case of the harmonic-oscillator-type interactions ($n = 0$), here the geometric links are missing in (8) and (10) thus each of the N bodies moves independently of the others as if they were bounded to the center of mass of the N -body system by springs of elastic constants given by $K_{ij}^{eff} = \alpha_i m$, so they do not need to be on the vertices of a regular geometric figure. The condition (12) has the same form as that one found in Ref. [14] showing the necessary condition for the Jacobi coordinates $\rho = \mathbf{r}_2 - \mathbf{r}_1$ and $\lambda = \mathbf{r}_3 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$ be conducive to the separation of the variables in the three-body problem with harmonic interactions.

It is worth pointing out that other solutions to the N -body problem with central forces can be obtained if the condition (8) is not imposed for all i, j and k but only for j and k which label the nearest particles of i -th one. This last condition is less restrictive than that one presented in Ref. [7] which requires N bodies of similar masses subject to similar forces moving on the vertices of a regular N -sided polygon. Taking into account that weaker condition one is conducted to all sorts of regular geometric figures. Regular polygons can rotate about an axis perpendicular to the plane with a common angular velocity while they change their sides at the same pace. For regular polyhedrons, though, expansions or contractions are the only permitted movements.

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- [6] See Ref. 3, Sec. 6.5, p 402
- [7] Sommerfeld A 1953 *Mechanics Lectures on Theoretical Physics* vol 1, Chap. V, § 32, pp 174-180 (New York: Academic Press)
- [8] This case is left as an exercise in Ref. 2, Sec. 12.8, p 494.
- [9] See Ref. 3, Sec. 6.5, pp 402-404
- [10] The existence of this solution is mentioned in Ref. 7, Chap. V, § 32, p 180
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